AIAA 81-4122

Parameter Plane Analysis for Large Flexible Spacecraft

Bernard A. Asner Jr.*
University of Dallas, Irving, Texas
and
Sherman M. Seltzer†
Control Dynamics Company, Huntsville, Ala.

The parameter plane technique for stability analysis and system design has been modified to obviate the sometimes cumbersome (particularly for high-order systems) derivation of the system characteristic equation. Instead, it is proposed to implement this technique by manipulating the matrices of the original equations. The objective is to keep the concepts simple and leave the tedious tasks to the computer. Using this technique, it is shown how a system may be designed by pole placement. This is implemented by mapping selected contours from the complex plane onto a selected two-parameter plane. Two primary advantages over existing frequency-domain techniques are: 1) system dynamics are portrayed in terms of two parameters, rather than a single parameter, thereby increasing the power of the designer's tool; and 2) the original system equations may be used without having to obtain the open- or closed-loop transfer functions or the characteristic equation.

Introduction

THERE are many instances when the engineer grapples with an analytical tool that is nice in mathematical formulation, but when it comes time to apply this tool, unpacking the notation becomes a formidable task. One has only to recall how easily the mathematician solves the linear equation $Ax=b$ by writing $x=A^{-1}b$ and then attempts a practical solution for a moderately large matrix A. An observation, made clear by many, but nevertheless worth repeating, is that A^{-1} is seldom required, and one should seek x by other methods. The parameter plane technique for determining stability regions of a linear time-invariant system is another example where the mathematical formulation is rather simple. One merely constructs the characteristic polynomial of the plant and treats this equation as a function of three variables. The system designer selects two of the variables as parameters of interest and lets the third be the indeterminate of the characteristic polynomial.

One may map selected contours from the complex domain onto the selected two-parameter plane. These contours may represent stability boundaries or contours of some selected measure of "relative" stability. In either event, one is mapping the s-domain locations of the characteristic equation roots onto the selected two-parameter plane. The primary advantage over extant frequency-domain techniques is that system dynamics are portrayed in terms of two parameters, rather than a single parameter (or gain), thereby doubling (in the sense of parameter variation portrayal) the power of the designer's analytical tool.

The parameter plane method provides an analytical and computational tool for use in the design and investigation of the dynamics of controlled systems, and, as presently formulated, the technique requires the system under investigation be described by its characteristic equation. The method is based on both linear and nonlinear techniques that are amply described in the first few pages of Siljak's monograph on the subject.[1] References 2-4 describe extensions of the method to sampled-data systems.

The history of the continuous time-domain version of the parameter plane technique is well documented in Ref. 1. Briefly, the technique has its origin with I.A. Vishnegradsky. He developed and used the first version of the parameter plane technique to portray system stability and transient characteristics of a third-order system on a two-parameter plane. In 1949, Yu I. Neimark generalized Vishnegradsky's approach to permit the decomposition of a two-parameter domain (D) describing an nth order system into stable and unstable regions.[5] The technique assumed the availability of the characteristic equation and was called D-decomposition. During the period 1959-1966, D. Mitrovic, founder of a Belgrade group of automatic control, extended the method to enable the analyst to relate the system's variable parameters to the system response by using the last two coefficients of the characteristic equation. Beginning in 1964, D.D. Siljak, then a student of Mitrovic, generalized the method and called it the parameter plane method.[6] His method permitted the analyst to select an arbitrary pair of characteristic equation coefficients (or parameters appearing within the coefficients) and to portray both graphically and analytically the dependence of the system response upon the selected parameters. The method was extended subsequently by Siljak and others to encompass a host of related problems. In 1967, J. George modified the D-composition method to enable the portrayal of the stability region in a multiparameter space.[7] George and Siljak also showed how to portray contours of relative stability.

A portion of the history that has not been reported on previously (with one exception to be noted) is the control system work conducted by the German rocket scientists in the early 1940's. Dr. W. Haeussermann and others applied the D-decomposition technique to the design of the V-2 rocket, following Dr. Haeussermann's earlier (pre-World War II) application to the control of an underwater torpedo. This work was not published in the open literature because of national security constraints. When the group came to the United States, Dr. Haeussermann and his associates continued to apply the method to U.S. Army missiles (and later to space vehicles). Again, national security (this time, another nation!) precluded publication in the open literature until 1957.[8]

In 1966 and later, the parameter space method has been applied to the design of missiles,[9] launch vehicles,[10,11] aircraft,[12] and satellite controllers,[13,14] including systems containing one or two nonlinearities,[15] the analysis of the dynamic effects of the nonlinear "solid friction" (Dahl) model for systems with ball bearings, such as control moment

Presented at 2nd VPI&SU/AIAA Symp. on Dyn. & Con. of Large Flexible Spacecraft, Blacksburg, Va., June 21-23, 1979; recd. Nov. 26, 1979; rev. recd. July 12, 1980.

*Associate Professor, Dept. of Mathematics.
†Consulting Engineer. Associate Fellow AIAA.

gyroscopes and reaction wheels[16]; and the specification by the system designer of the dynamic structural flexibility constraints to the structural designer.[17]

The major shortcoming of the method has been the requirement to have available the system characteristic equation. In this paper, it is pointed out that a direct route to the equations used to determine stability boundaries is available by using conventional computer techniques. It is proposed herein to implement the parameter plane analysis by manipulating the original system equations and, therefore, not requiring the cumbersome derivation of the characteristic equation. The objective of this technique is to keep the concepts simple and leave the tedious tasks to the computer. In cognizance that computer time and storage are important factors, some optimization of the algorithm is performed. However, in this initial work, sparse matrix theory is not considered. Hence, the order of the system will be restricted to be less than 100 or so for medium-sized computers.

In this paper, we substitute the parameter plane method for other frequency-domain techniques. It does not provide a substitute for state space techniques but can be used to augment the state space approach. One way would be to prescribe the stability region within which state feedback gains must lie. Another way would be to use the state space approach to determine the optimal values of gains. The parameter space could then be used to determine the effects of variations of selected system parameters, thereby permitting deduction of system sensitivity or robustness.[18]

Tutorial

One may use the parameter plane to portray stability boundaries, thereby indicating the region (or regions) of stability in terms of the two selected parameters. This is performed by letting $s=j\omega$ in the characteristic equation. This leads to two simultaneous algebraic equations for the selected parameters, which may be denoted as α and β. Solving these equations for α and β leads to a solution(s) in terms of ω and the remaining system variables. These variables are considered to be numerically given, and values of ω are assigned sequentially from zero to infinity. The solution for each assigned value of ω generates a point on the stability boundary. As ω varies, these point solutions will generate a curve that will separate the plane into disjoint sets such that (α,β) in the interior of each set implies the number of roots where positive real parts will remain unchanged. This boundary, the "dynamic boundary," is associated with complex conjugate roots of the characteristic equation.

Another stability boundary, the static boundary, is associated with a mapping of the $s=0$ point from the complex domain onto the two-parameter plane. This contour (if it exists) is found by selecting values of (α,β) which force the smallest coefficient of the characteristic equation to be identically zero. The static boundary normally is found by letting $s=0$ in the characteristic equation. This boundary depends on the other system variables, but is independent of ω.

A third stability boundary, the boundary at infinity, is found by setting the largest coefficient of the characteristic equal to zero and solving the resulting equation for (α,β).

In determining regions in the parameter plane, one must be alert for the possible existence of singular cases where the Jacobian of the two algebraic equations is equal to zero and a manifold of solutions may exist. In most situations this does not occur and a local solution is guaranteed by the inverse mapping theorem. (see Ref. 1, p. 383).

Contours associated with the dynamics of the system (sometimes termed "relative" stability contours) may also be displayed. This might be implemented by letting $s=\sigma+j\omega$ and requiring all roots of the characteristic equation to lie to the left of a prescribed value of σ. One may set $s=\sigma+j\omega$ in the characteristic equation and proceed as with the mapping of the stability boundary associated with complex conjugate roots, except that each contour is now to be associated with the prescribed value of σ. Similarly, a real root contour may be found by setting $s=\delta\neq 0$ in the characteristic equation.

Perhaps a more useful set of contours for the structural dynamicist is that associated with prescribed values of the damping ratio ζ rather than the damping factor σ. This is accomplished by letting $\sigma=-\zeta\omega_n$ and $\omega=\omega_n\sqrt{1-\zeta^2}$, where ζ is the damping factor and ω_n is the undamped natural frequency, and, proceeding as above, to find ζ contours with ω_n as a parameter. This process can be repeated for as many values of ζ as deemed necessary. If the contours for the real roots (generated by letting $s=\delta$) are superimposed on the ζ contours, then one can read off all the roots of the characteristic equation for each value of α and β. The designer can now prescribe system dynamics by pole placement or assignment.

As an example of the parameter plane technique, we consider a simplified model of a spacecraft with a flexible appendage where it is desired to specify the controller gains to provide a desired response. The spacecraft model portrays the rotational dynamics in a single plane and comprises the main theme of Ref. 17.

Example 1

The model is described by the following equations of motion:

$$\varphi_e=\varphi_c-\varphi_a \tag{1}$$

$$\varphi_a=\varphi_r+\varphi_b \tag{2}$$

$$T_c=I\ddot{\varphi}_r \tag{3}$$

$$K_bT_c=\ddot{\varphi}_b+2\zeta_b\omega_b\dot{\varphi}_b+\omega_b^2\varphi_b \tag{4}$$

$$T_c=K_p\varphi_e+K_i\int\varphi_e\mathrm{d}t-K_d\varphi_r \tag{5}$$

where Eqs. (3) and (4) represent the plant dynamics, Eq. (5) represents the onboard controller, and Eqs. (1) and (2) represent the system kinematics. Angles represented by symbols φ_c, φ_a, φ_r, and φ_b are the commanded input attitude, the actual spacecraft attitude, the contribution to vehicle attitude by rigid-body dynamics, and the contribution to vehicle attitude by flexible-body dynamics, respectively. For this elementary example, only the first bending mode is considered. Also, a linear combination of rigid- and flexible-body dynamics is assumed in Eq. (2), while the position error φ_e is defined by Eq. (1). The overall vehicle moment of inertia for the assumed planar rotational motion is symbolized by I, and the commanded torque by T_c. The assumed first modal damping ratio and natural frequency are symbolized by ζ_b and ω_b, respectively. The first modal gain is shown as K_b. The design problem at hand is to select numerical values for control gains K_p, K_i, and K_d that will provide a desired response of the system. The parameter plane technique will be applied to show how this may be accomplished using a pole placement (assignment) approach.

The equations may be recast in the Laplace domain and written as the vector matrix equation

$$A(s)X(s)=d \tag{6}$$

where the vector $X(s)$ is the Laplace transform of $(\varphi_r\varphi_aT_c\varphi_b)^T$ and the matrix A is defined as

$$A=\begin{bmatrix}1 & 0 & 0 & 1\\ 0 & -(K_ds^2+K_ps+K_i) & -s & 0\\ Is^2 & 0 & -1 & 0\\ 0 & 0 & K_b & -(s^2+2\zeta_b\omega_bs+\omega_b^2)\end{bmatrix} \tag{7}$$

and d is a vector of initial values and reference input. Let the control gains be normalized as

$$a_v = K_v / I \qquad (v = b,p,i,d) \tag{8}$$

Then the characteristic equation is

$$\Delta(s) = a_p(\mathrm{d}s) + a_i d + a_d(\mathrm{d}s^2) + s^3(d - a_b s^2) = 0 \tag{9}$$

where

$$d = (a_b + 1)s^2 + 2\zeta_b \omega_b s + \omega_b^2$$

Selecting the parameters as $\alpha = a_p$ and $\beta = a_d$, the technique, as described in this section, may now be applied to Eq. (9). The following values were used as typical: $\zeta_b = 0.005$, $\omega_b = 6$ rad/s, $I = 23{,}185$ kg/m^2, and $K_b = 4.3131 \times 10^{-6}$ (ks/m^2)$^{-1}$, (updated Space Telescope values from Ref. 17). A value of $a_i = 0.04$ (rad/s^3)$^{-1}$ is selected in this example. Using these

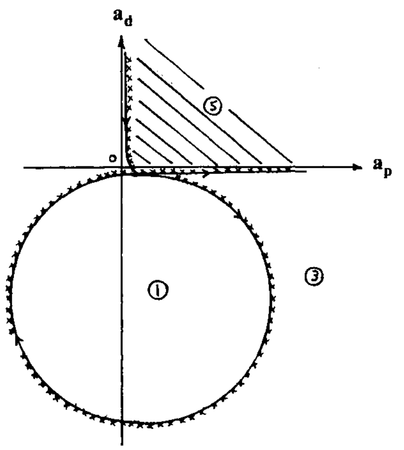

Fig. 1 Sketch of simplified spacecraft stability region—example 1. Legend: cross-hatched area in first quadrant = stable region; × = stability contour; circled numbers = number of stable roots in region (total of five roots in system).

values and the parameter plane technique, the values of control gains a_p and a_d that bound a stable region are displayed in Fig. 1. It was obtained by letting $s = j\omega$ (i.e., by letting $\zeta = 0$), and varying ω in value from zero to infinity. Due to the wide dispersion of numbers, Fig. 1 is not shown to scale.

Now the process may be repeated for selected values of ζ, yielding the constant ζ contours which are shown as functions of ω in Fig. 2. Each point on a ζ contour represents the values of a pair of complex conjugate roots belonging to the fifth-order (in this case) characteristic equation. If one also sets s equal to real values (denoted on Fig. 2 as δ), one may obtain the contours associated with each real root. Thus, each point on Fig. 2 is intersected by two contours (each representing a pair of complex conjugate roots) and a single δ contour (representing one root).

Suppose it is desired to prescribe a damping ratio of $\zeta = 1/\sqrt{2}$. That means values of a_p and a_d must be chosen so that one remains on the $1/\sqrt{2}$ contour. If one chooses "Design Pt. #1," then we have one pair of (dominant) roots with $\zeta = 1/\sqrt{2}$ and $\omega_n = 3.3$ rad/s, one pair of roots with $\zeta = 0.04$ and $\omega_n = 5.9$ rad/s, and a real root with $\delta = -0.0045$. Design experience might lead one to place the natural frequencies of the complex roots farther apart. One then moves down the $1/\sqrt{2}$ contour (i.e., toward the origin) to "Design Pt. #2," where the dominant pair of roots has $\zeta = 1/\sqrt{2}$ and $\omega_n = 1.4$ rad/s; one pair of roots has $\zeta = 0.02$ and $\omega_n = 6.0$ rad/s; and the real root is at $\delta = -0.023$. This might be judged to be a sufficient frequency separation. Hence, one has designed the control system by pole selection using an $a_p = 2$ and $a_d = 1.8$. To ensure that the response is sufficiently good, one now may simulate the original system with these values to obtain the response in the time domain. If performance is unsatisfactory, various values of a_i may be tried. Alternately, one can select a second parameter plane, using a_i as one of the new parameters, and use the two-parameter plane plots in conjunction with each other. With a modest amount of

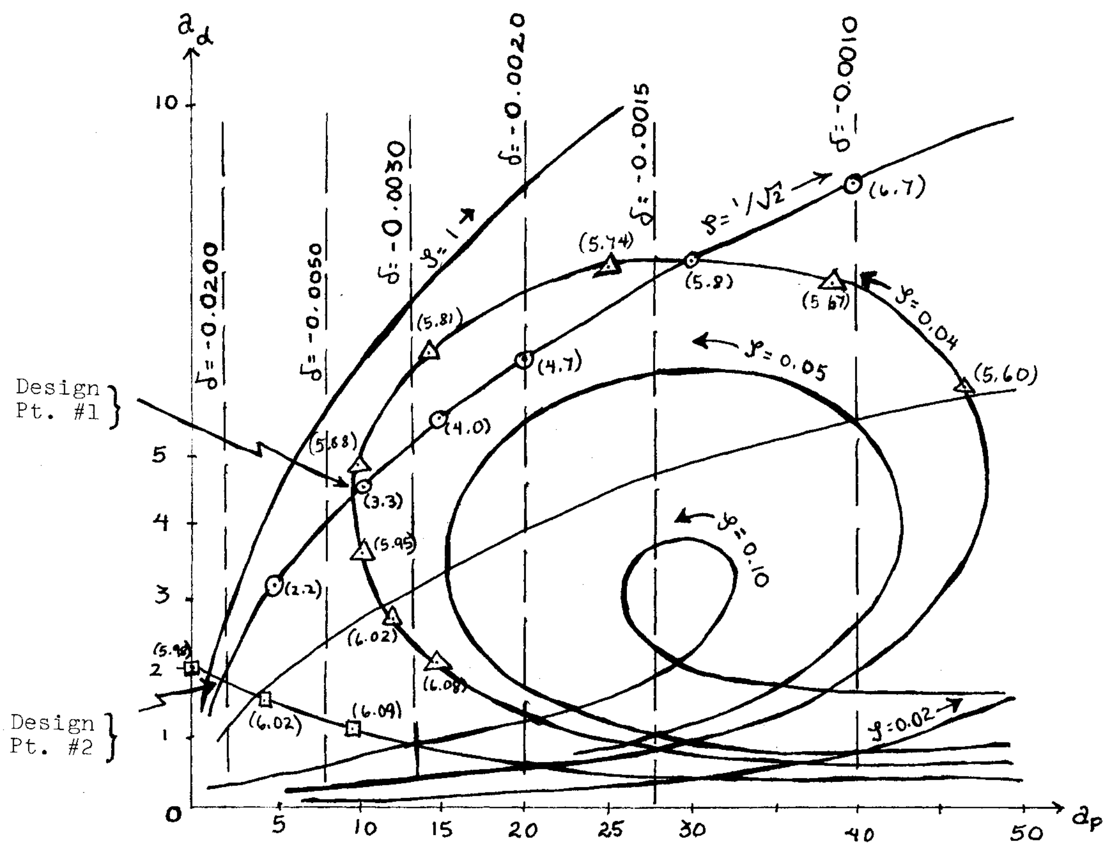

Fig. 2 ζ and δ contours—example 1. Legend: numbers in parentheses indicate values of ω; arrows indicate direction of increasing ω; solid lines indicate contours of constant ζ; dashed lines indicate values of real roots.

ingenuity, this latter technique can be used to develop a three-parameter space in terms of all three PID (a_p, a_i, a_d) gains.

Analysis for Large Systems

If one considers an enhanced model by including more states, then the order of the system equations will increase. For the parameter plane technique, one usually assumes the existence of the characteristic equation. For a large order system, however, the explicit derivation of the characteristic equation can be a formidable task and one should try another approach. In this section, we present an alternate route that works on the original system of equations and leaves the tedious computational tasks to the computer.

Let A denote a matrix of order n with complex entries a_{ij}. Assume the elements a_{ij} are functions of the complex number s and the real parameters α and β:

$$a_{ij} = a_{ij}(s;\alpha,\beta) \qquad (1 \le i,j \le n) \tag{10}$$

The elements of A are constructed by taking the Laplace transform of a linear time-invariant system of equations. Formally, s^k replaces the kth derivative and s^{-k} replaces the k-fold integral in the original equations. Since the zeros of the characteristic equation are identical to those of $\det(A) = 0$, one seeks values of s, α, and β which make the determinant equal to zero, i.e.,

$$\Delta(s,\alpha,\beta) = 0 \tag{11}$$

and these values define a curve in Euclidean space R^2. In particular, if $s = j\omega$, the surface will separate R^2 into disjoint sets such that (α,β) in the interior of each set implies the number of roots with positive real parts will remain unchanged.

It is assumed that the parameters occur in only two elements of A. There are two basic reasons for this last restriction. The first is the desire to present a procedure which does not obstruct the central thought and, if needed, can be readily modified. The second reason stems from the observation that many mathematical models fit nicely into this category. For example, a control engineer trying to adjust the gains K_p, K_i, and K_d in a controller, or the generalized masses and natural frequencies of bending modes, will note that these parameters fit the second requirement.

Suppose the parameters α, β are located at $i = p$, $j = q$ and at $i = u$, $j = v$, respectively, so that

$$a_{pq} = a_{pq}(s;\alpha,\beta) \qquad \text{and} \qquad a_{uv} = a_{uv}(s;\alpha,\beta) \tag{12}$$

Using elementary operations of determinants, one may rewrite Eq. (11) as

$$a_{pq}a_{uv}B(s) + a_{pq}C(s) + a_{uv}D(s) + E(s) = 0 \qquad (p \ne u) \tag{13}$$

where $E(s)$ is the determinant of the matrix formed from A by letting $a_{pq} = a_{uv} = 0$; $D(s)$ is the determinant of the matrix formed from A by letting $a_{pq} = 0$ and replacing row u with e_v^T (e_j^T is a row vector with 1 in position j and zeros elsewhere); $C(s)$ is the determinant of the matrix formed from A by letting $a_{uv} = 0$ and replacing row p with e_q^T; and $B(s)$ is the determinant of the matrix formed from A by replacing rows p and u with e_q^T and e_v^T, respectively. In case $p = u$ (identified as Case II in the sequel), then $B(s) = 0$ and one obtains

$$a_{pq}C(s) + a_{pv}D(s) + E(s) = 0 \qquad (p = u) \tag{14}$$

Using the relations in Eq. (12) with Eq. (13) or (14), substituting $s = \sigma + j\omega$, and separating the real and imaginary parts of the resulting equation produces two simultaneous equations that can be solved for α and β.

The success or failure of the decomposition depends upon the ability to calculate the coefficients in Eq. (13) [or Eq. (14) if $p = u$]. It is proposed to calculate an equivalent set of coefficients by modifying the Gaussian elimination procedure with partial pivoting. This will be accomplished for all four determinants by essentially only one pass through the Gaussian scheme. It will be described for the two cases where $p \ne u$ (case I) and $p = u$ (case II).

Case I: $(p \ne u)$

By a proper permutation of the system equations, it is possible to reposition the element containing α at $i = n-1$, $j = n-1$ and the element containing β at $i = n$, $j = n$. If one denotes $a_{n-1,\, n-1} = a$ and $a_{nn} = b$, then the decomposition of $\det(A($ [i.e., Δ of Eq. (11)] leading to Eq. (13) can be pictured as follows:

$$\begin{vmatrix} \times & \times \ldots \times & \times \\ \times & \times \ldots \times & \times \\ \hline / & / \ldots a & / \\ / & / \ldots / & b \end{vmatrix}$$

$$= ab \begin{vmatrix} \times & \times \ldots \times & \times \\ \times & \times \ldots \times & \times \\ \hline 0 & 0 \ldots 1 & 0 \\ 0 & 0 \ldots 0 & 1 \end{vmatrix} + a \begin{vmatrix} \times & \times \ldots \times & \times \\ \times & \times \ldots \times & \times \\ \hline 0 & 0 \ldots 1 & 0 \\ / & / \ldots / & 0 \end{vmatrix}$$

$$+ b \begin{vmatrix} \times & \times \ldots \times & \times \\ \times & \times \ldots \times & \times \\ \hline / & / \ldots 0 & / \\ 0 & 0 \ldots 0 & 1 \end{vmatrix} + \begin{vmatrix} \times & \times \ldots \times & \times \\ \times & \times \ldots \times & \times \\ \hline / & / \ldots 0 & / \\ / & / \ldots / & 0 \end{vmatrix}$$

$$\triangleq abB(s) + aC(s) + bD(s) + E(s) \tag{15}$$

The $\times$'s are used to denote elements in the first $n-2$ rows, and the slashes (/) denote elements (other than zeros, ones, a's or b's) in the remaining last two rows in the determinants.

For computational reasons, one divides Eq. (15) by $B(s) \ne 0$ and computes the ratios of the resulting equation

$$ab + a[C(s)/B(s)] + b[D(s)/B(s)] + E(s)/B(s) = 0 \tag{16}$$

If Gaussian elimination with partial pivoting is performed on the first $n-2$ rows of any of the ratios in Eq. (16), one arrives at the following scheme for each ratio:

$$\begin{vmatrix} \text{-----------------------} \\ 0 \;\text{-----------------} \\ \qquad\text{-------------} \\ / \; / \; / \; / \; / \; / \; / \; / \; / \\ / \; / \; / \; / \; / \; / \; / \; / \; / \\ \text{-----------------------} \\ \text{-----------------------} \\ 0 \;\text{-----------------} \\ \qquad\text{-------------} \\ 0\;0\;0 \;\; \text{--------------}1\;0 \\ 0\;0\;0 \;\; \text{--------------}0\;1 \end{vmatrix}$$

One observes that the work required to reach this stage is identical for both the numerator and denominator and need be accomplished only once. To finish the algorithm, one merely replaces the last two rows of the numerator by those of C, D, and E and continues with Gaussian elimination. Further, one observes that the ratio of the determinants is independent of the first $n-2$ diagonal elements, implying that the required answer will be the product of the last two diagonal elements. Also note that one does not need to keep track of the number of row interchanges.

Summary of Algorithm ($p \neq u$)

1) Set $a_{n-1,n-1} = a_{nn} = 0$ and perform a Gaussian elimination with pivoting on the first $n-2$ rows. Continue with Gaussian elimination, but without pivoting. Then record

$$D(s)/B(s) = a_{n-1,n-1} \quad \text{and} \quad E(s)/B(s) = a_{n-1,n-1}a_{nn}$$

2) Set $a_{n-1,n-1} = 1$, $a_{n-1,n} = 0$, then restore the original nth row with $a_{nn} = 0$ in the current nth row. Restart the Gaussian scheme and take advantage of the fact that the matrix in the first $n-1$ rows is already in triangular form. Record $C(s)/B(s) = a_{nn}$.

Case II ($p = u$)

As the parameters α,β occur on the same row, a permutation of the original system equations will reposition both elements on the last row with $a_{n-1,n-1} = a$ and $a_{nn} = b$. Arguments leading to Eq. (16) also yield the equation

$$aC(s) + bD(s) + E(s) = 0 \tag{17}$$

For a parameter plane to exist, one must have $C(s) \neq 0$ or $D(s) \neq 0$. Without loss of generality, one may assume $D(s) \neq 0$ and consider the equation

$$a[C(s)/D(s)] + b + [E(s)/D(s)] = 0 \qquad (D(s) \neq 0) \tag{18}$$

Similar arguments, as in Case I, lead to the following summary.

Summary of Algorithm ($p = u$)

1) Set $a_{n,n-1} = a_{nn} = 0$ and perform a Gaussian elimination with pivoting on the first $n-1$ rows. Record $C(s)/D(s) = -a_{n-1,n}/a_{n-1,n-1}$.

2) Continue the Gaussian scheme for the last row and record $E(s)/D(s) = a_{nn}$.

Remark: In the authors' computer code, the near singularities, $B(s) \approx 0$ in Eq. (16) or $D(s) \approx 0$ in Eq. (18), are dealt with by selecting another term for the division.

Examples

To demonstrate the utility of the foregoing modified parameter plane technique, it will be applied to two examples. The first example of this section will be a continuation of example 1 in the Tutorial section. The second example is one of higher (15th-order) complexity. Comparing examples 2 and 3 shows the complexity of the technique is scarcely altered as the order of the system increases.

Example 2

Suppose, after satisfactory control gains are determined for the model of example 1, it is desired to investigate system robustness with respect to modal characteristics a_b, ω_b, or ζ_b. (Typically, numerical values of bending characteristics are poorly known prior to spacecraft flight.) In this example modal parameters a_b and ω_b are selected, although any pair may be used. Control gain values chosen in example 1 are used. If the control gains are normalized using Eq. (8), then the matrix of Eq. (7) can be rewritten as

$$\Delta = \begin{vmatrix} 1 & -1 & 0 & 1 \\ 0 & -(a_d s^2 + a_p s + a_i) & -s & 0 \\ s^2 & 0 & -1 & 0 \\ 0 & 0 & a_b & -(s^2 + 2\zeta_b\omega_b s + \omega_b^2) \end{vmatrix} = 0 \tag{19}$$

The algorithm as described in the last section may now be applied to the determinant of Eq. (19). The results for a typical set of constants are depicted on Figs. 3 and 4. Figure 3 shows the results of the analysis for $0 \leq \omega < \infty$ and, due to the wide dispersion of numbers, is not shown to scale. However, Fig. 4 is to scale and shows the stable region that is of practical interest to the problem at hand.

For pedagogical reasons, the remainder of this example follows the spirit of the developed computer program. Expanding the determinant of Eq. (19) yields an equation of the form [see Eq. (17)]

$$aC(s) + bD(s) + E(s) = 0 \tag{20}$$

where

$$a = a_b \qquad b = -(s^2 + 2\zeta_b\omega_b s + \omega_b^2) \qquad c = a_d s^2 + a_p s + a_i$$

$$C(s) = \begin{vmatrix} 1 & -1 & 0 & 1 \\ 0 & -c & -s & 0 \\ s^2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} \tag{21a}$$

$$D(s) = \begin{vmatrix} 1 & -1 & 0 & 1 \\ 0 & -c & -s & 0 \\ s^2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \tag{21b}$$

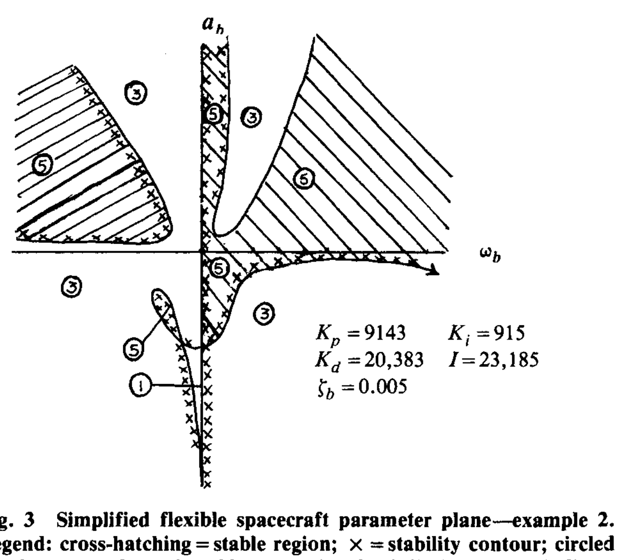

Fig. 3 Simplified flexible spacecraft parameter plane—example 2. Legend: cross-hatching = stable region; × = stability contour; circled numbers = number of stable roots (total of five); arrows indicate direction of increasing ω.

$$E(s)=\begin{vmatrix} 1 & -1 & 0 & 1 \\ 0 & -c & -s & 0 \\ s^2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} \tag{21c}$$

Dividing Eq. (20) by $D(s)$, and noting that $E(s)=0$, one obtains the equivalent expression $a[C(s)/D(s)]+b=0$. An application of the Gaussian elimination technique to the first three rows of the ratio in the brackets yields

$$C/D=\frac{\begin{vmatrix} 1 & -1 & 0 & 1 \\ 0 & -c & -s & 0 \\ 0 & 0 & -1-s^3/c & -s^2 \\ 0 & 0 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 & 1 \\ 0 & -c & -s & 0 \\ 0 & 0 & -1-s^3/c & -s^2 \\ 0 & 0 & 0 & 1 \end{vmatrix}}$$

$$=\frac{(1)(-c)(s^2)}{(1)(-c)(-1-s^3/c)}=-s^2/(1+s^3/c) \tag{22}$$

This last ratio, together with the definitions of a and b, results in

$$s^2+2\zeta_b\omega_b s+\omega_b^2+s^2/(1+s^3/c)a_b=0 \tag{23}$$

which is equivalent to the characteristic equation of the original system of equations. With the exception of ω_b and a_b, every term in the last equation can be treated as a complex number that has been determined numerically. Thus, one has two equations to solve for the two unknowns $\alpha=\omega_b$ and $\beta=a_b$.

Example 3

This example is included to indicate how the parameter plane technique can be applied to a high-order (21 states in the case at hand) system. The model used is one currently in use by NASA and represents the roll dynamics of the Solar Electric Propulsion System (SEPS).[19] The rigid-body plant dynamics are represented by the expression

$$T=I\ddot{\phi}_r \tag{24}$$

where T represents the applied torque, I the spacecraft moment of inertia, and φ_r the rigid-body contribution to spacecraft attitude. As in the previous examples, rotational planar motion has been assumed. The flexible-body plant dynamics are represented by nine normal modes of vibration. The kth mode of vibration may be expressed as

$$\ddot{\varphi}_k+2\zeta_k\omega_k\dot{\varphi}_k+\omega_k^2\varphi_k=K_kT \tag{25}$$

where φ_k represents the kth modes' contribution to the spacecraft attitude; and ζ_k, ω_k, and K_k represent the damping ratio, natural frequency, and gain of the kth bending mode, respectively (see Table 1 for numerical values). It is assumed that the spacecraft attitude is composed of a linear combination of the rigid-body and flexible-body normal mode contributions; i.e.,

$$\varphi=\varphi_r+\Sigma\varphi_k \tag{26}$$

where, in the case at hand, the sum ranges from one to nine. The attitude sensor dynamics are depicted as a first-order lag, i.e.,

$$\tau_s\dot{\varphi}_s+\varphi_s=\varphi \tag{27}$$

where τ_s represents the sensor time constant and φ_s the sensed attitude of the spacecraft. The controller in this case takes the form of a lead-lag shaping network,

$$\tau_2\dot{\psi}+\psi=\tau_1\dot{\phi}_s+\varphi_s \tag{28}$$

where τ_1 and τ_2 represent the network time constants and ψ represents the output of the network. The angle between the thrust vector (F) attitude and the vehicle centerline is denoted as β. The relation of β to ψ is shown by the equation $\beta=-K\psi$, where K is a scalar. If the angle β is small, the relationship of

Table 1 Normal mode characteristics (example 3)

j	ζ_j	ω_j	θ_i	$K_j=\theta_j^2/175$
1	0.01	0.915	0.0922	4.858 E-05
2	0.01	2.73	−0.1534	−1.345 E-04
3	0.01	5.41	0.1546	1.366 E-04
4	0.01	9.37	0.1195	8.160 E-04
5	0.01	14.9	0.08865	4.491 E-04
6	0.01	21.9	−0.0683	−2.666 E-05
7	0.01	30.3	0.052	1.545 E-05
8	0.01	44.1	−0.041	−9.606 E-06
9	0.01	57.7	0.0403	9.281 E-06

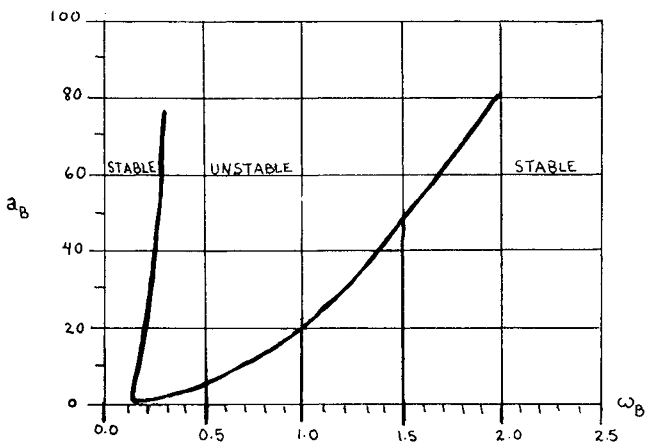

Fig. 4 Stability region in first quadrant—example 2.

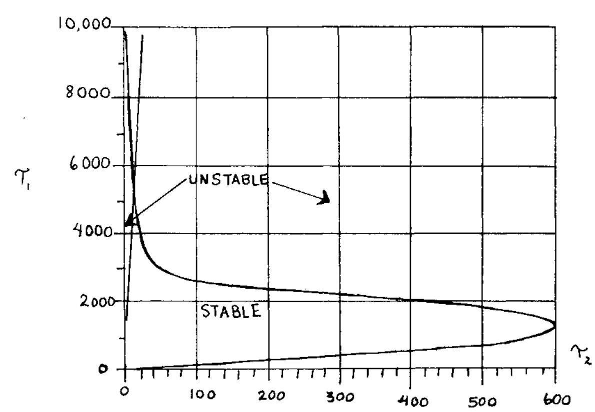

Fig. 5 Stability region with 9 bending modes—example 3.

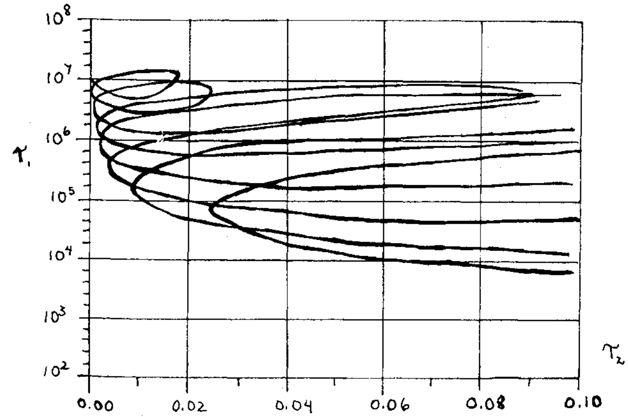

Fig. 6 Bending effects of 9 modes—example 3.

β to T may be expressed as

$$T = F X_{cm} \beta \tag{29}$$

where x_{cm} is the distance between the spacecraft center of mass and the point of application of F.

Using the techniques of this paper and the developed computer program, a parameter plane for the stability boundary of τ_1 vs τ_2 was generated. Calculations were performed on a Sigma 7 where 4.3 min of CPU time were utilized to generate 2767 data points. The region of interest is shown as Fig. 5. The influence of the bending modes occurs for $\tau_1 \gg 1000$ and for $0 \geq \tau_2 \leq 20$. Because it is of academic interest, a portion of this latter region is shown as Fig. 6. In generating the data, the following numerical values were used: $I = 187{,}574$ kg/m^2, $F = 1$N, $\tau_s = 10$ s, $K = 15.7$, $x_{cm} = 0.5$ m.

Conclusions

The primary objective of this paper has been to provide a means of obviating the sometimes cumbersome necessity for deriving the characteristic equation in order to apply the parameter plane technique. Because the description of this tool has been scattered in various journals, the essentials are repeated herein in order to provide a compact, cohesive paper. It has been shown how the technique can be applied to portray stability regions in terms of two variable parameters, such as control gains. It has been indicated how the technique may be used to portray a stable region in terms of more than two selected parameters.

It is shown how the parameter plane technique may be used for both analysis and system design using pole placement. It should be apparent to the reader how the technique can be used to indicate robustness with respect to selected (by the designer) system parameters.

The technique is limited to systems that can be described by linear equations, although it has been used in systems containing nonlinear elements. Presently under investigation is the extension of the techniques described herein to sampled-data systems.

Acknowledgment

Portions of this research were accomplished under an NRC Resident Research Associateship at the Systems Dynamics Laboratory, NASA George C. Marshall Space Flight Center, Alabama.

References

[1] Siljak, D.D., *Nonlinear Systems,* Wiley, New York, 1969.

[2] Siljak, D.D., "Analysis and Synthesis of Feedback Control Systems in the Parameter Plane, Part II—Sampled-Data Systems," *IEEE Transactions, Part II, Applications and Industry,* Vol. 83, Nov. 1964, pp. 458-466.

[3] Seltzer, S.M., "Sampled-Data Control System Design in the Parameter Plane," *Proceedings of the 8th Annual Allerton Conference on Circuits and System Theory,* 1970, pp. 454-463.

[4] Seltzer, S.M., "Enhancing Simulation Efficiency with Analytical Tools," *Computers in Electrical Engineering,* Vol. 2, 1975, pp. 35-44.

[5] Neimark, Yu I., "D-decomposition of the Space of Quasipolynomials (On the Stability of Linearized Distributed Systems)" (in Russian), *Applied Mathematics and Mechanics,* Vol. 13, No. 4, 1949, pp. 349-380.

[6] Siljak, D.D., "Generalization of Mitrovic's Method," *IEEE Transactions, Part II, Applications and Industry,* Vol. 83, Sept. 1964, pp. 314-320.

[7] George, J.H., "On Parameter Stability Regions for Several Parameters Using Frequency Response," *IEEE Transactions on Automatic Control,* Vol. AC-12, No. 2, April 1967, pp. 197-200.

[8] Haeussermann, W., "Stability Areas of Missile Control Systems," *Jet Propulsion,* July 1957, pp. 787-795.

[9] Seltzer, S.M., "Parameter Plane Techniques Applied to Tactical Missile Control System Design," *Proceedings of Eighth Annual IEEE Region III Conference,* Huntsville, Ala., Nov. 1969.

[10] Seltzer, S.M., "An Acceleration Vector Attitude Control System," *Journal of Spacecraft and Rockets,* Vol. 5, June 1968, pp. 666-671.

[11] Seltzer, S.M. and Siljak, D.D., "Absolute Stability Analysis of Attitude Control Systems for Large Space Boosters," *Proceedings of the IV IFAC Symposium on Automatic Control in Space,* Dubrovnik, Yugoslavia, Sept. 1971.

[12] Huang, J.T. and Seltzer, S.M., "Parameter Plane Stability Analysis of a Flight Vehicle," AIAA Paper 70-983, AIAA Guidance, Control, and Flight Mechanics Conference, Santa Barbara, Calif, Aug. 1970.

[13] Asner, B., Schweitzer, G., and Seltzer, S.M., "Attitude Control of a Spinning Skylab," *Journal of Spacecraft and Rockets,* Vol. 10, March 1973, pp. 200-207.

[14] Seltzer, S.M., "A Satellite Digital Controller," *Journal of Spacecraft and Rockets,* Vol. 14, Aug. 1977, pp. 509-511.

[15] Seltzer, S.M., "Analysis of a Control System Containing Two Nonlinearities," *International Journal of Control,* 1971, Vol. 13, No. 6, pp. 1057-1064.

[16] Seltzer, S.M., "Large Space Telescope Oscillations Induced by CMG Friction," *Journal of Spacecraft and Rockets,* Vol. 12, Feb. 1975, pp. 96-105.

[17] Seltzer, S.M. and Shelton, H.L., "Specification of Spacecraft Flexible Appendage Rigidity," *Journal of Guidance and Control,* Vol. 1, Nov.-Dec. 1978, pp. 427-432.

[18] Seltzer, S.M., Asner, B.A., and Jackson, R.L., "Parameter Plane Analysis for Large Scale Systems," AIAA Paper 80-1790, *Proceedings of AIAA Guidance and Control Conference,* Danvers, Mass., Aug. 1980, pp. 414-419.

[19] "SEPS Role in the Development and Exploration of Space. A User's Manual," Boeing Aerospace Company, NASA Marshall Space Flight Center, D180-19783-4, July 1976.